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LETTER TO THE EDITOR

A coupled Korteweg–de Vries equation with dispersion

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Abstract. A dispersive system describing a vector multiplet interacting with the Korteweg–de Vries field is shown to be a member of a bi-Hamiltonian integrable hierarchy.

An integrable system can be sometimes represented as a sub- or factor-system of a larger integrable system. Such an extension may or may not have all the features of the original system. The simplest integrable system, the KdV equation, provides an instructive case. It has a number of integrable *two-component* extensions, of the form

$$\dot{u} = u_{xxx} + 6uu_x - 12vv_x, \qquad \dot{v} = -2(v_{xxx} + 3uv_x), \tag{1}$$

$$\dot{u} = u_{xxx} + 6uu_x + 2vv_x, \qquad \dot{v} = 2(uv)_x. \tag{2}$$

The system (1) was introduced in Hirota and Satsuma (1981); it is associated with the affine Lie algebra $C_2^{(1)}$, and has only *one* Hamiltonian structure extending the second Hamiltonian structure of the KdV equation (Wilson 1982). The system (2) has two Hamiltonian structures extending each of the two Hamiltonian structures of the KdV equation (Ito 1982); however, it has no *dispersion* in the v equation. This defect can be removed.

Consider the following *multicomponent* extension of the KdV equation:

$$\dot{u} = -u_{xxx} + 6uu_x + 2\boldsymbol{v}^t\boldsymbol{v}_x + \boldsymbol{c}^t\boldsymbol{v}_{xx}, \qquad \dot{\boldsymbol{v}} = (2u\boldsymbol{v})_x - u_{xx}\boldsymbol{c}, \tag{3}$$

where $\boldsymbol{v} = (v_1, \ldots, v_N)^t$, and $\boldsymbol{c} = (c_1, \ldots, c_N)^t$ is a constant (column) vector. For $\boldsymbol{c} = 0$, $\boldsymbol{v} = v$, (3) collapses into (2) (after an inessential rescaling). Let us see that (3) is a bi-Hamiltonian system with an infinite number of conservation laws (CLS). Let us rewrite (3) in the form

$$\begin{pmatrix} \dot{u} \\ \dot{\boldsymbol{v}} \end{pmatrix} = B^2(\delta H_1) = B^1(\delta H_2), \tag{4}$$

$$H_0 = \tfrac{1}{2}u_2, \qquad H_1 = \tfrac{1}{2}(u^2 + \boldsymbol{v}^t\boldsymbol{v}), \qquad H_2 = u^3 + \tfrac{1}{2}u^{(1)2} + u(\boldsymbol{v}^t\boldsymbol{v} + \boldsymbol{c}^t\boldsymbol{v}^{(1)}), \tag{5}$$

$$B^1 = \mathrm{diag}(\partial, \partial\mathbb{1}), \qquad B^2 = 2B + b_\omega + b_\nu, \tag{6}$$

$$B = \left(\begin{array}{c|c} u\partial + \partial u & \boldsymbol{v}^t\partial \\ \hline \partial\boldsymbol{v} & \boldsymbol{0} \end{array}\right), \qquad b_\omega = \mathrm{diag}(-\partial^3, \boldsymbol{0}), \qquad b_\nu = \left(\begin{array}{c|c} 0 & \boldsymbol{c}^t \\ \hline -\boldsymbol{c} & \boldsymbol{0} \end{array}\right)\partial^2, \tag{7}$$

where $\partial = \partial/\partial x$, $(\,\cdot\,)^k = \partial^k(\,\cdot\,)$, and $\delta H = (\delta H/\delta u,\ \delta H/\delta \boldsymbol{v}^t)^t$ is the column vector of functional derivatives of H. The matrix B^1 is skewsymmetric constant-coefficient and, thus, is Hamiltonian (see ch I in Manin (1979)). Let K be a commutative algebra with a derivation $\partial: K \to K$ (say, $K = C^\infty(\mathbb{R}^1)$). Let $D(K)$ be K considered as a Lie

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algebra ('vector fields on $\mathbb{R}^1$') with the commutator

$$[X, Y] = XY^{(1)} - X^{(1)}Y, \qquad X, Y \in K. \tag{8}$$

Let $y = D(K) \ltimes K^N$ be the semidirect product Lie algebra with the commutator

$$\left[\begin{pmatrix} X \\ \boldsymbol{p} \end{pmatrix}, \begin{pmatrix} Y \\ \boldsymbol{q} \end{pmatrix}\right] = \begin{pmatrix} XY^{(1)} - X^{(1)}Y \\ X\boldsymbol{q}^{(1)} - Y\boldsymbol{p}^{(1)} \end{pmatrix}, \qquad X, Y \in K, \qquad \boldsymbol{p}, \boldsymbol{q} \in K^N. \tag{9}$$

Then B in (7) is the natural Hamiltonian form associated to ('the dual space of') the Lie algebra y. In addition, consider bilinear forms ω and ν on y whose associated operators b_ω and b_ν are given in (7):

$$\omega\left(\begin{pmatrix} X \\ \boldsymbol{p} \end{pmatrix}, \begin{pmatrix} Y \\ \boldsymbol{q} \end{pmatrix}\right) = -XY^{(3)}, \qquad \nu\left(\begin{pmatrix} X \\ \boldsymbol{p} \end{pmatrix}, \begin{pmatrix} Y \\ \boldsymbol{q} \end{pmatrix}\right) = X\boldsymbol{c}^t\boldsymbol{q}^{(2)} - \boldsymbol{p}^t\boldsymbol{c}Y^{(2)}. \tag{10}$$

It is easy to see that ω and ν are (generalised) two-cocycles on y. Therefore, the matrix B^2 is Hamiltonian (see ch VIII in Kupershmidt (1985).)

Now let us show that the bi-Hamiltonian definition

$$B^1(\delta H_{n+1}) = B^2(\delta H_n) \tag{11}$$

can be iterated for all n. It will imply that we have a whole bi-Hamiltonian hierarchy with a common infinite set of CLS. Denote

$$a_n = \delta H_n/\delta u, \; \boldsymbol{b}_n = \delta H_n/\delta \boldsymbol{v}, \qquad F_n = \delta H_n.$$

Rewriting (4) in long hand, we obtain

$$\begin{pmatrix} a_{n+1}^{(1)} \\ \boldsymbol{b}_{n+1}^{(1)} \end{pmatrix} = \begin{pmatrix} [2(u\partial + \partial u) - \partial^3](a_n) + 2\boldsymbol{v}^t\boldsymbol{b}_n^{(1)} + \boldsymbol{c}^t\boldsymbol{b}_n^{(2)} \\ (2a_n\boldsymbol{v} - a_n^{(1)}\boldsymbol{c})^{(1)} \end{pmatrix} = B^2\begin{pmatrix} a_n \\ \boldsymbol{b}_n \end{pmatrix}, \tag{12}$$

so that we can set

$$\boldsymbol{b}_{n+1} = 2a_n\boldsymbol{v} - a_n^{(1)}\boldsymbol{c}. \tag{13}$$

Also, using (11), we get

$$F_m^t B^2(F_n) \sim -F_n^t B^2(F_m) = -F_n^t B^1(F_{m+1}) \sim F_{m+1}^t B^1(F_n) = F_{m+1}^t B^2(F_{n-1}), \tag{14}$$

where $a \sim b$ means: $(a - b) \in \operatorname{Im} \partial$. Hence, we obtain from (14) that

$$F_m^t B^2(F_n) \sim 0. \tag{15}$$

In particular, taking $m = 0$ in (15), so that $F_0 = (\frac{1}{2}, \boldsymbol{0})^t$, we obtain

$$[2(u\partial + \partial u) - \partial^3](a_n) + 2\boldsymbol{v}^t\boldsymbol{b}_n^{(1)} + \boldsymbol{c}^t\boldsymbol{b}_n^{(2)} \sim 0, \tag{16}$$

which implies that we can find a_{n+1} in (12) for every n. It remains to show that, for each n thus obtained, vector $F_n = (a_n, \boldsymbol{b}_n^t)^t$ is a vector of functional derivatives of some H_n. This is equivalent (Manin 1979, Kupershmidt 1980) to showing that the Fréchet derivative $D(F_n)$ is symmetric, where

$$D(F_n) = \begin{pmatrix} D_u(a_n) & D_{\boldsymbol{v}}(a_n) \\ D_u(\boldsymbol{b}_n) & D_{\boldsymbol{v}}(\boldsymbol{b}_n) \end{pmatrix} \tag{17}$$

with $D_u(a) = \Sigma(\partial a/\partial u^{(m)})\partial^m$, etc. We show that

$$D_n^\dagger = D_n, \qquad D_n := D(F_n), \tag{18}$$

by induction on n, the cases $n=0$, 1, 2 being obviously satisfied. Taking the Fréchet derivative of (12), we get

$$B^1 D_{n+1} = B^2 D_n + 2G_n \tag{19n}$$

$$G_n = \begin{pmatrix} a_n^{(1)} + \partial a_n & \boldsymbol{b}_n^{(1)t} \\ \boldsymbol{0} & \partial a_n \mathbb{1} \end{pmatrix}. \tag{20}$$

Applying to (19n) from the right the operator B^1, and subtracting from the result the composition of (19$n-1$) and B^2, we find

$$B^1 D_{n+1} B^1 = (B^2 D_n B^1 + B^1 D_n B^2) - B^2 D_{n-1} B^2 + 2\bar{G}_n, \tag{21}$$

$$\bar{G}_n := G_n B^1 - G_{n-1} B^2. \tag{22}$$

Since $B^1 D_{n+1} B^1$ is symmetric when D_{n+1} is, to make the induction step we have to show that $\bar{G}_n$ is symmetric, and this is a straightforward calculation based on the following identities

$$\{(a^{(1)} + \partial a)[2(u\partial + \partial u) - \partial^3] - (2au^{(1)} + 4a^{(1)}u - a^{(3)}\partial\} \qquad \text{is symmetric} \tag{23}$$

$$[\boldsymbol{b}^{(1)t}(2\partial \boldsymbol{v} - \boldsymbol{c}\partial^2) - (2\boldsymbol{v}^t \boldsymbol{b}^{(1)} + \boldsymbol{c}^t \boldsymbol{b}^{(2)})\partial] \qquad \text{is symmetric} \tag{24}$$

$$[\partial a(2\partial \boldsymbol{v}^t - \boldsymbol{c}^t \partial^2)]^\dagger = (a^{(1)} + \partial a)(2\boldsymbol{v}\partial + \boldsymbol{c}\partial^2) - (2a\boldsymbol{v} - a^{(1)}\boldsymbol{c})^{(1)}\partial. \tag{25}$$

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